a) Bestimmen Sie das reelle Integral $\int_0^{2\pi} \frac{\sin x}{5+3\sin x}\,dx$

Hinweis: $\int_0^{2\pi} \frac{\sin x}{5+3\sin x}\,dx = a + b\pi$, a und b müssen bestimmt werden

$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

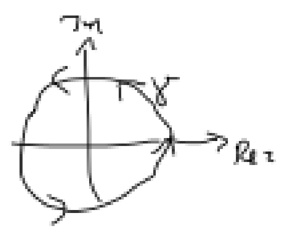
$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$$\int_0^{2\pi} \frac{\sin x}{5+3\sin x}\,dx = \int_0^{2\pi} \frac{\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)}{5+3\cdot\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)}\,dz = \int_0^{2\pi} \underbrace{\frac{\frac{1}{2i}\left(e^{it}-\frac{1}{e^{it}}\right)}{5+\frac{3}{2}\frac{1}{i}\left(e^{it}-e^{-it}\right)}}_{f(\gamma(t))} \underbrace{i\cdot e^{it}}_{\gamma'(t)}\,dt = \int_0^{2\pi} f(\underbrace{\gamma(t)}_{=e^{it}})\,\underbrace{\gamma'(t)}_{=i\cdot e^{it}}\,dt$$

$\gamma : [0, 2\pi] \to \mathbb{C}$

$t \mapsto e^{it}$

$\gamma'(t) = (e^{it})' = i\cdot e^{it}$



$$\tilde{f}(z) = \frac{\frac{1}{2i}\left(z-\frac{1}{z}\right)}{5+\frac{3}{2}\frac{1}{i}\left(z-\frac{1}{z}\right)}\cdot\frac{1}{i\cdot z} = \frac{\frac{1}{2i}\frac{z^2-1}{z}}{5z+\frac{3}{2}\frac{1}{i}(z^2-1)}\cdot\frac{1}{i} = \frac{1}{2}\cdot\frac{1}{i^2}\,\frac{z^2-1}{z\cdot\left(5z+\frac{3}{2i}(z^2-1)\right)}$$

$$= -\frac{1}{2}\cdot\frac{z^2-1}{z\cdot\left(5z+\frac{3}{2i}(z^2-1)\right)}$$

$$= \int_\gamma \tilde{f}(z)\,dz = -\frac{1}{2}\int_\gamma \underbrace{\frac{z^2-1}{z\cdot\left(5z+\frac{3}{2i}(z^2-1)\right)}}_{f(z)}\,dz = -\frac{1}{2}\cdot 2\pi i \sum_{a\in\mathbb{C}} \mathrm{Ind}_\gamma(a)\cdot \mathrm{Res}(a,f)$$

$$= -\frac{1}{2}\cdot 2\pi i\cdot \underbrace{\mathrm{Ind}_\gamma(0)}_{=1}\cdot \mathrm{Res}(0,f)$$

$z=0$ ist Pol 1. Ordnung

$5z + \frac{3}{2i}(z^2-1) = 0$

$\frac{3}{2i}z^2 + 5z - \frac{3}{2i} = 0 \quad \Big| \cdot \frac{2}{3}i$

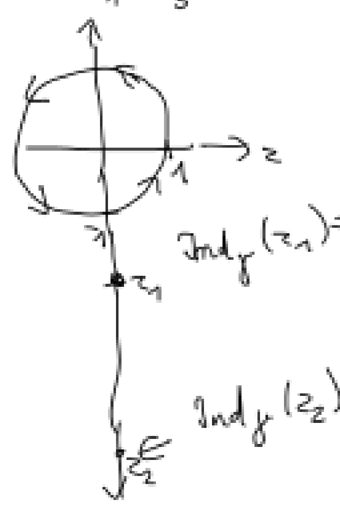
$z^2 + \frac{10}{3}iz - 1 = 0$

$$z_{1/2} = -\frac{10}{3}i \pm \sqrt{\left(\frac{\frac{10}{3}i}{2}\right)^2 + 1}$$

$$= -\frac{10}{3}i \pm \sqrt{-\frac{25}{9}+1} = -\frac{10}{3}i \pm \sqrt{-\frac{16}{9}}$$

$$= -\frac{10}{3}i \pm \frac{4}{3}i$$

$z_1 = -\frac{6}{3}i = -2i \quad z_2 = -\frac{14}{3}i$



$\mathrm{Ind}_\gamma(z_1) = 0$

$\mathrm{Ind}_\gamma(z_2) = 0$

$$f(z) = \frac{\overbrace{z^2-1}^{g(z)}}{\underbrace{z\cdot\left(5z+\frac{3}{2i}(z^2-1)\right)}_{h(z)}}$$

Pol n-ter Ordnung

$$\mathrm{Res}(f, a) = \frac{g^{(n-1)}(a)}{(n-1)!} \quad \text{wobei } g(z) = (z-a)^n\cdot f(z)$$

$$\mathrm{Res}(0,f) = \frac{g^{(1-1)}(0)}{(1-1)!} = \frac{g(0)}{\underbrace{0!}_{=1}}$$

$$= \frac{0^2-1}{5\cdot 0+\frac{3}{2i}(0^2-1)} = \frac{-1}{\frac{3}{2i}\cdot(-1)}$$

$$= \frac{2}{3}i$$

$$= -\frac{1}{2}\cdot 2\pi i\cdot\frac{2}{3}i$$

$$= \frac{2}{3}\pi$$

$$\frac{1}{(m-1)!}\cdot\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m\cdot f(z)\right)$$

$m = 1$

$$= \frac{1}{(1-1)!}\cdot\lim_{z\to 0}\cancel{z}\cdot\underbrace{f(z)}_{\frac{z^2-1}{\cancel{z}\cdot\left(5z+\frac{3}{2i}(z^2-1)\right)}} = \frac{0^2-1}{5\cdot 0+\frac{3}{2i}(0^2-1)}$$