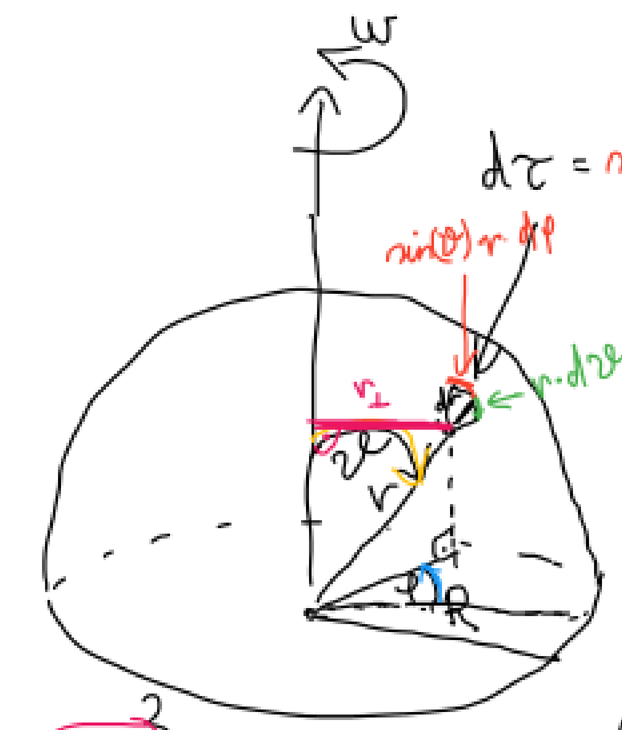
$d\tau = \sin\vartheta\, r\, d\varphi\, r\, d\vartheta\, dr$
$= \sin\vartheta \cdot r^2\, d\varphi\, d\vartheta\, dr$

b)



$\sin\vartheta = \frac{r_\perp}{r}$

$r_\perp = \sin\vartheta \cdot r$

Aufgabe 1:

Bestimmen Sie die Trägheitsmomente der folgenden Körper durch Berechnen des Volumenintegrals:

(a) Hohlzylinder mit der Höhe H, Innenradius R_i, Außenradius R_a, Dichte ρ (die Rotationsachse ist gleich der Zylinderachse).

(b) Halbkugel mit dem Radius R und der Dichte ρ (die Rotationsachse ist die Symmetrieachse).

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx$$

$$\int_a^b f(x)\cdot g'(x) = \left[f(x)\cdot g(x)\right]_a^b - \int_a^b f'(x)\cdot g(x)\,dx$$

$$\int_0^{\frac{\pi}{2}} \sin^3(x)\,dx = \int_0^{\frac{\pi}{2}} \underbrace{\sin^2(x)}_{f(x)}\cdot \underbrace{\sin(x)}_{g'(x)}\,dx =$$

$$= \underbrace{\left[\sin^2(x)\cdot(-1)\cos(x)\right]_0^{\frac{\pi}{2}}}_{=0} - \int_0^{\frac{\pi}{2}} 2\sin(x)\cdot\cos(x)\cdot(-1)\cdot\cos(x)\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \sin(x)\cdot\cos^2(x)\,dx =$$

$$= 2\int_0^{\frac{\pi}{2}} \sin(x)\cdot(1-\sin^2(x))\,dx$$

$$= 2\cdot\left(\int_0^{\frac{\pi}{2}} \sin(x)\,dx - \underbrace{\int_0^{\frac{\pi}{2}} \sin^3(x)\,dx}_{\text{Phönix aus der Asche}}\right) \Rightarrow \int_0^{\frac{\pi}{2}} \sin^3(x)\,dx = 2\int_0^{\frac{\pi}{2}}\sin(x)\,dx - 2\int_0^{\frac{\pi}{2}}\sin^3(x)\,dx$$

$(f(g(x)))' = f'(g(x))\cdot g'(x)$

$((\underbrace{\sin(x)}_{g(x)})^2)' = 2\cdot\sin(x)\cdot(\sin(x))' = 2\cdot\sin(x)\cdot\cos(x)$

$f(x) = x^2$

$$\Rightarrow 3\int_0^{\frac{\pi}{2}} \sin^3(x)\,dx = 2\underbrace{\int_0^{\frac{\pi}{2}} \sin(x)\,dx}_{=\left[-\cos(x)\right]_0^{\frac{\pi}{2}} = 1}$$

1 : 3

$$\Rightarrow \boxed{\int_0^{\frac{\pi}{2}} \sin^3(x) = \frac{2}{3}}$$

$$J = \int_{V_{\text{Halbkugel}}} m(\vec r)\cdot r_\perp(\vec r)^2\, d\tau(\vec r) =$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho\,(\sin\vartheta\cdot r)^2 \sin\vartheta\cdot r^2\, d\vartheta\, d\varphi\, dr =$$

$$= \rho\cdot\int_0^R r^4\,dr \cdot \int_0^{2\pi} 1\,d\varphi \cdot \int_0^{\frac{\pi}{2}} \sin^3\vartheta\, d\vartheta =$$

$$= \rho\cdot\frac{1}{5}R^5\cdot 2\pi\cdot\frac{2}{3} = \frac{4}{2}\rho\cdot\frac{4}{3}\cdot\pi\cdot R^3\cdot R^2\cdot\frac{1}{5} = 2\cdot M\cdot\frac{1}{5}R^2 = \frac{2}{5}MR^2$$

$M = V\cdot\rho$
$= \frac{1}{2}V_{\text{Kugel}}\cdot\rho$
$= \frac{1}{2}\cdot\frac{4}{3}\pi R^3\cdot\rho$

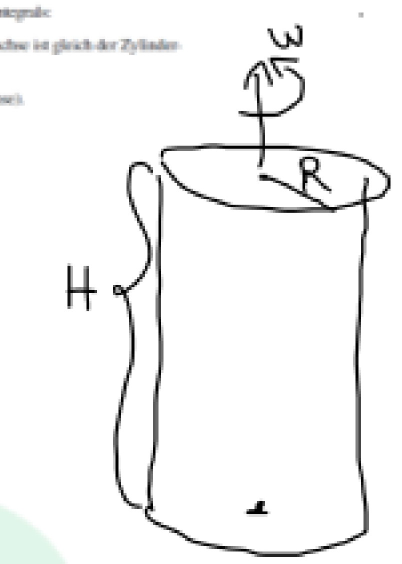
$$\boxed{J_{\text{Halbkugel}} = \frac{2}{5}MR^2}$$

$d\tau = r^2\cdot\sin\vartheta\, d\varphi\, d\vartheta\, dr$

$$J = \int_{V_{\text{Halbkugel}}} \cancel{m(\vec r)}^{\,\rho}\, r_\perp(\vec r)^2\, d\tau(\vec r) =$$

$dm = \rho\cdot d\tau$

$$\int_{V_{\text{Hk}}} r_\perp(\vec r)^2 \qquad \underbrace{\overset{\rho}{m(\vec r)}\, d\tau(\vec r)}_{\text{Masse eines Volumen}}$$



$J = \frac{1}{2}MR^2$

$$J = \int_{V_{\text{Zyl}}} m(\vec r)\cdot r_\perp(\vec r)^2\, d\tau(\vec r) =$$

$$= \int_0^H \int_0^R \int_0^{2\pi} \rho\cdot r^2 \cdot r\,d\varphi\, dr\, dz$$

$$J = \int_0^H \int_{R_i}^{R_a} \int_0^{2\pi} \rho\, r^2\, r\, d\varphi\, dr\, dz$$

$$= \rho\cdot\int_0^H \int_{R_i}^{R_a} r^3 \underbrace{\int_0^{2\pi} 1\,d\varphi}_{=2\pi}\, dz =$$

$$= \rho\cdot 2\pi\cdot\int_0^H \underbrace{\int_{R_i}^{R_a} r^3\,dr}_{=\left[\frac{1}{4}r^4\right]_{R_i}^{R_a} = \frac{1}{4}(R_a^4 - R_i^4)}\, dz$$

$$= \rho\cdot 2\pi\cdot\frac{1}{4}(R_a^4 - R_i^4)\cdot\underbrace{\int_0^H 1\,dz}_{=H}$$

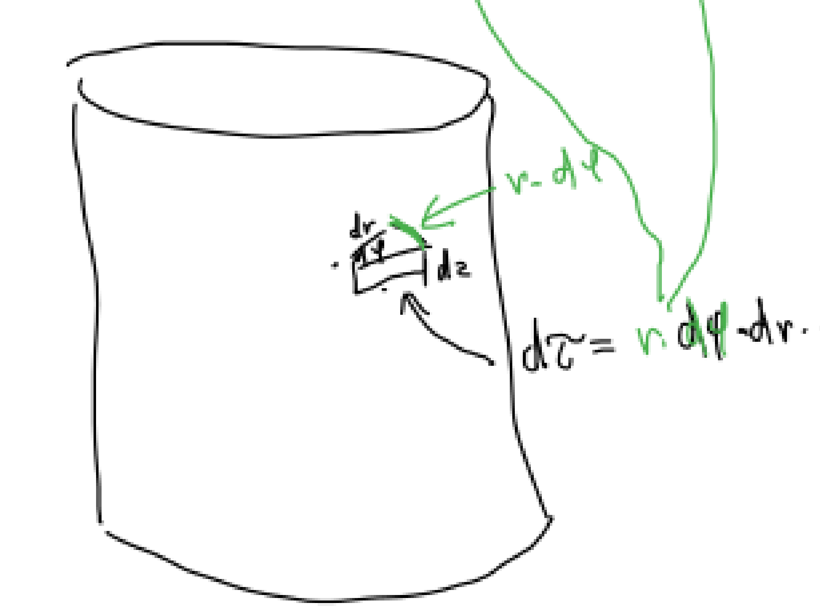
$$= \rho H 2\pi\cdot\frac{1}{4}\cdot(R_a^2 - R_i^2)\cdot(R_a^2 + R_i^2)$$

$(a+b)(a-b) =$

$$= M\cdot\frac{2}{4}\cdot(R_a^2 + R_i^2)$$

$$= \frac{1}{2}\cdot M\cdot(R_a^2 + R_i^2)$$

$M = V\cdot\rho$
$= (\pi R_a^2\cdot H - \pi R_i^2\cdot H)\cdot\rho$
$= \pi\cdot H\cdot(R_a^2 - R_i^2)\cdot\rho$



$d\tau = r\,d\varphi\,dr\,dz$