$\hat{A}$ operator

$\forall \psi, \phi$:

$\langle \hat{A}^{\dagger}\psi, \phi \rangle = \langle \psi, \hat{A}\phi \rangle$

Problem 6.1: Orthogonality of bound state solutions

Two states $\varphi_1(x)$ and $\varphi_2(x)$ are bound state solutions (real functions) of the Schrödinger equation, with energy eigenvalues E_1 and E_2 respectively, where $E_1 \neq E_2$. Show that $\varphi_1(x)$ and $\varphi_2(x)$ are orthogonal. (*4 points*)

$$H\varphi_1 = E_1 \cdot \varphi_1$$
$$H\varphi_2 = E_2 \cdot \varphi_2$$

A operator

δ eigenvalue of A :⇔

$\exists \psi \neq 0$ such that

$A \cdot \psi = \delta \cdot \psi$

eigenvector

Jeder hermitesche Operator $\hat{A}$ hat nur reelle Eigenwerte.

H ist hermitesch,

d.h. $H = H^{\dagger}$, d.h.

$\forall \psi, \phi$:

$\langle H\psi, \phi \rangle = \langle \psi, H\phi \rangle$

Zu zeigen: $\langle \varphi_1, \varphi_2 \rangle = 0$

$$E_1\langle \varphi_1, \varphi_2 \rangle = E_1^{*} \cdot \langle \varphi_1, \varphi_2 \rangle = \langle E_1\varphi_1, \varphi_2 \rangle = \langle H\varphi_1, \varphi_2 \rangle \overset{H \text{ hermitesch}}{=} \langle \varphi_1, H\varphi_2 \rangle =$$
$$= \langle \varphi_1, E_2\varphi_2 \rangle =$$
$$= E_2 \cdot \langle \varphi_1, \varphi_2 \rangle$$

$E_1 \in \mathbb{R}$

$\boxed{\forall a \in \mathbb{R}: a^{*} = a}$

$$\Rightarrow E_1 \cdot \langle \varphi_1, \varphi_2 \rangle = E_2 \cdot \langle \varphi_1, \varphi_2 \rangle \quad \big| - E_2 \cdot \langle \varphi_1, \varphi_2 \rangle$$

$$\underbrace{(E_1 - E_2)}_{\neq 0} \cdot \langle \varphi_1, \varphi_2 \rangle = 0$$

$$\Rightarrow \langle \varphi_1, \varphi_2 \rangle = 0$$

$\alpha \in \mathbb{C}$

$\langle \alpha\varphi_1, \varphi_2 \rangle = \alpha^{*}\langle \varphi_1, \varphi_2 \rangle$

$\langle \varphi_1, \alpha\varphi_2 \rangle = \alpha\langle \varphi_1, \varphi_2 \rangle$

$$\langle \alpha\varphi_1, \varphi_2 \rangle = \int_{-\infty}^{\infty} (\alpha\varphi_1)^{*}\varphi_2 \, dx = \int_{-\infty}^{\infty} \alpha^{*}\varphi_1^{*}\varphi_2 = \alpha^{*}\langle \varphi_1, \varphi_2 \rangle$$

Beweis: Sei $\delta \in \mathbb{C}$ ein Eigenwert von $\hat{A}$, d.h. wir finden ein $\psi \neq 0$ sodass

$\hat{A}\psi = \delta \cdot \psi$.

$$\delta^{*}\langle \psi, \psi \rangle = \langle \delta\psi, \psi \rangle = \langle \hat{A}\psi, \psi \rangle \overset{\hat{A} \text{ hermitesch}}{=} \langle \psi, \hat{A}\psi \rangle = \langle \psi, \delta\psi \rangle =$$
$$= \delta\langle \psi, \psi \rangle$$

$$\Rightarrow \delta^{*}\langle \psi, \psi \rangle = \delta \cdot \langle \psi, \psi \rangle \quad \big| - \delta \cdot \langle \psi, \psi \rangle$$

$$(\delta^{*} - \delta) \cdot \underbrace{\langle \psi, \psi \rangle}_{\neq 0,\ \text{weil } \psi \neq 0} = 0$$

$$\delta^{*} - \delta = 0$$

$$\Rightarrow \delta^{*} = \delta \Rightarrow \delta \in \mathbb{R}$$

$\langle \cdot, \cdot \rangle$ ist positiv definit

$\langle \psi, \psi \rangle = 0 \Leftrightarrow \psi = 0$

Problem 6.2: Particle in an asymmetric infinite potential well

Consider a particle of mass m in a potential field given by

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & x < 0 \vee x > L. \end{cases}$$

At the initial time $t = 0$ the wave function of the particle is given by