Problem 3.3: Position and momentum operators

In one-dimensional space the coordinate operator $\hat{x}$ and the momentum operator $\hat{p}$ are defined as

$$\hat{x}\,\psi(x) = x\,\psi(x), \qquad \hat{p}\,\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x),$$

where $\psi(x)$ is a wave function. Calculate the following commutators:

a) (*4 points*) $[\hat{p}^2, \hat{x}^2]$ and $[\hat{p}^2, \hat{x}^{-1}]$,

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{p}^2, \hat{x}^2]\psi = \left(\hat{p}^2 \cdot \hat{x}^2 \psi - \hat{x}^2 \hat{p}^2 \psi\right) =$$

$$= -\hbar^2 \frac{\partial^2}{\partial x^2}(x^2\psi) - x^2 \cdot (-1) \cdot \hbar^2 \frac{\partial^2}{\partial x^2}\psi(x) =$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$= \hbar^2\left(-\frac{\partial^2}{\partial x^2}(x^2\psi(x)) + x^2 \frac{\partial^2}{\partial x^2}\psi(x)\right) =$$

$$= \hbar^2\left(-\frac{\partial}{\partial x}\left(2x \cdot \psi + x^2 \cdot \frac{\partial}{\partial x}\psi\right) + x^2 \frac{\partial^2}{\partial x^2}\psi\right)$$

$$= \hbar^2\left(-\left(2\psi + 2x\frac{\partial \psi}{\partial x} + 2x\frac{\partial}{\partial x}\psi + x^2\frac{\partial^2 \psi}{\partial x^2}\right) + x^2\frac{\partial^2}{\partial x^2}\psi\right)$$

$$= \hbar^2\left(-2\psi - 4x\frac{\partial \psi}{\partial x}\right) = -2\hbar^2\left(\psi + 2x\frac{\partial \psi}{\partial x}\right) =$$

$$= \left(-2\hbar^2\left(1 + 2x\frac{\partial}{\partial x}\right)\right)\psi$$

$$[\hat{p}^2, \hat{x}^2] = -2\hbar^2\left(1 + 2x\frac{\partial}{\partial x}\right)$$

$$\tilde{\ } \ldots = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{(k_0+k)^2}{2a}}$$

$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r},t)$	Freie Schrödinger-gleichung

$$i\cdot\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

$$\int a\cdot f(x)\,dx = a\cdot\int f(x)\,dx$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\psi}(k,t)\cdot e^{ikx}\,dk$$

$$\cancel{\frac{1}{\sqrt{2\pi}}}\, i\cdot\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\tilde{\psi}(k,t)\,e^{ikx}\,dk = -\cancel{\frac{1}{\sqrt{2\pi}}}\cdot\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\tilde{\psi}(k,t)\,e^{ikx}\,dk$$

$$i\cdot\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\tilde{\psi}(k,t)\cdot e^{ikx}\,dk = -\frac{\hbar}{2m}\int_{-\infty}^{\infty}\tilde{\psi}(k,t)\underbrace{\frac{\partial^2}{\partial x^2}e^{ikx}}_{-k^2e^{ikx}}\,dk$$

$$(e^{ikx})' = e^{ikx}\cdot ik$$
$$(e^{ikx})'' = -k^2e^{ikx}$$

$$\int_{-\infty}^{\infty}e^{ikx}\cdot\Big(\underbrace{i\cdot\frac{\partial}{\partial t}\tilde{\psi}(k,t) - \frac{\hbar}{2m}\tilde{\psi}(k,t)\,k^2}_{=0}\Big)\,dk = 0$$

$$\boxed{i\cdot\frac{\partial}{\partial t}\tilde{\psi}(k,t) - \frac{\hbar}{2m}\cdot k^2\,\tilde{\psi}(k,t) = 0}$$

$$\frac{\partial}{\partial t}\tilde{\psi} = \frac{\hbar}{2mi}k^2\tilde{\psi}$$

$$\tilde{\psi}(k,t) = C\cdot e^{\frac{\hbar}{2mi}k^2\cdot t}$$

$$\boxed{\tilde{\psi}(k,t) = \frac{1}{\sqrt[4]{\pi}}\cdot e^{-\frac{(k_0+k)^2}{2a}}\cdot e^{\frac{\hbar}{2mi}k^2t}}$$

$$\boxed{f'(x) = a\cdot f(x)},\quad f(0) = 100$$

$$f(x) = 100\cdot e^{ax}\qquad f(0) = C\cdot e^{a\cdot 0} = C\cdot 1 = C \overset{!}{=} 100$$

Probe: $f'(x) = C\cdot e^{ax}\cdot a = a\cdot C\cdot e^{ax} = a\cdot f(x)$ ✓

$$(e^{ax})' = a\cdot e^{ax}$$

$$\psi(k,0) = \frac{1}{\sqrt[4]{\pi}}e^{-\frac{(k_0+k)^2}{2a}}$$

$$7\cdot f'(x) = 2\cdot f(x),\quad f(0) = 3$$
$$f'(x) = \tfrac{2}{7}f(x)$$
$$f(x) = \underset{=3}{C}\cdot e^{\frac{2}{7}\cdot x} = 3\cdot e^{\frac{2}{7}x}\qquad f(0) = C\cdot e^{\frac{2}{7}\cdot 0} = C\cdot 1 \overset{!}{=} 3$$

Probe: $f'(x) = C\cdot\frac{2}{7}\cdot e^{\frac{2}{7}x} = \frac{2}{7}\cdot f(x)$ ✓

c) Show that the time-dependent wave packet $\psi(x,t)$ has the expression *(2 points)*

$$\psi(x,t) = \frac{1}{\sqrt{a\sqrt{\pi}(1+it/\tau)}}\exp\left(-\frac{(x-v_0t)^2}{2a^2(1+it/\tau)} + i(k_0x-\omega_0t)\right),$$

where

$$\omega_0 = \frac{\hbar k_0^2}{2m},\quad v_0 = \frac{\hbar k_0}{m},\quad \tau = \frac{ma^2}{\hbar}.$$

$$\int e^x\,dx = e^x$$

$$\int e^{-k^2}\,dk = ?$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty}\tilde{\psi}(k,t)\cdot e^{ikx}\,dk =$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt[4]{\pi}}\int_{-\infty}^{\infty}e^{-\frac{(k_0+k)^2}{2a}+\frac{\hbar}{2mi}k^2t+ikx}\,dk$$

quadratische Ergänzung und Gauß-Integral

$$\int_{-\infty}^{\infty}e^{-\delta(x^2-c)}\,dx = \sqrt{\frac{\pi}{\delta}}\quad\forall c\in\mathbb{C},\ \delta>0$$

$$(e^x)^2 = e^{2x}$$

$$\int_0^1 x\cdot e^{-x^2} = \left[-\frac{1}{2}\,e^{-x^2}\right]_0^1$$

d) *(To be discussed in class (2 points))* The average position of a particle at time t is defined as

$$\langle x\rangle_t = \int_{-\infty}^{\infty}x|\psi(x,t)|^2dx.$$

Show that

$$\langle x\rangle_t = v_0t.$$

$$\int_a^b|\psi(x,t)|^2dx =$$

W'keit, dass Teilchen zwischen a und b

$$\langle x\rangle_t = \int_{-\infty}^{\infty}x\cdot|\psi(x,t)|^2dx = \frac{1}{a\sqrt{\pi}(1+it/\tau)}\cdot\int_{-\infty}^{\infty}x\cdot e^{-\frac{(x-v_0t)^2}{a^2(1+it/\tau)}+2i(k_0x-\omega_0t)}\,dx =$$