Problem 1.1: Dirac δ-function

The Dirac δ-function δ(x) is defined through the formula

$$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x) = f(0).$$

Calculate the Fourier transform of the δ-function. Express δ(x) as an integral. What was the inverse Fourier transform of 1. (2 points)

The Fourier transform of a function f(x) is defined as [illegible]

The original function can be recovered by the inverse Fourier transformation [illegible]

$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \delta(x) \cdot \underbrace{e^{-ikx}}_{f(x)} dx = \frac{1}{\sqrt{2\pi}} \cdot f(0) = \frac{1}{\sqrt{2\pi}} \cdot e^0 = \frac{1}{\sqrt{2\pi}}$$

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

Problem 1.2: Fourier transforms of given functions

Calculate the Fourier transforms of the following functions

a)

$$f(x) = \begin{cases} f_0, & x \in [-x_0, x_0] \\ 0, & \text{otherwise,} \end{cases} \quad \text{(2 points)}$$

b)

$$f(x) = f_0 \exp(-x^2/x_0^2), \quad \text{(2 points)}$$

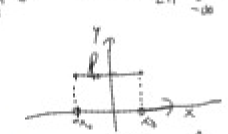
c)

$$f(x) = f_0 \exp(-|x|/x_0), \quad \text{(2 points)}$$

d)

$$f(x) = \frac{f_0}{x^2 + x_0^2}, \quad \text{(2 points)}$$

where $x_0 > 0$ and $f_0 > 0$ are two constants.



$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-x_0}^{x_0} f_0 \cdot e^{-ikx} dx =$$

$$\left(\frac{1}{a}e^{ax}\right)' = \frac{1}{a} a \cdot e^{ax} = e^{ax} \qquad \int e^x dx = e^x \qquad \int e^{ax} dx = \frac{1}{a} e^{ax}$$

$k \neq 0$:

$$= \frac{f_0}{\sqrt{2\pi}} \left[-\frac{1}{ik} e^{-ikx}\right]_{-x_0}^{x_0} = \frac{f_0 \cdot i}{k \cdot \sqrt{2\pi}} \left(e^{-ikx_0} - e^{ikx_0}\right) = \frac{f_0\, i \cdot 2i}{k \cdot \sqrt{2\pi}} \cdot \frac{e^{-ikx_0} - e^{ikx_0}}{2i} = \frac{f_0 \cdot i \cdot 2i}{k \cdot \sqrt{2\pi}} \sin(-kx_0) =$$

$$= \frac{-2f_0}{k\sqrt{2\pi}} \cdot (-1) \sin(kx_0) = \frac{2f_0}{k\sqrt{2\pi}} \sin(kx_0)$$

$\sin(-x) = -\sin(x)$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$k = 0$:

$$\tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-x_0}^{x_0} f_0\, dx = \frac{f_0}{\sqrt{2\pi}} [x]_{-x_0}^{x_0} = \frac{f_0}{\sqrt{2\pi}} (x_0 - (-x_0)) = \frac{f_0 \cdot 2x_0}{\sqrt{2\pi}}$$

$$\tilde{f}(k) = \begin{cases} \frac{f_0 2x_0}{\sqrt{2\pi}} & \text{falls } k = 0 \\ \frac{2f_0}{k\sqrt{2\pi}} \sin(kx_0) & \text{falls } k \neq 0 \end{cases}$$

b)

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_0\, e^{-\frac{x^2}{x_0^2}} \cdot e^{-ikx} dx = \frac{f_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{x_0^2} - ikx} dx$$

$$\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i \qquad e^a \cdot e^b = e^{a+b}$$

$\delta > 0$

$$\int_{-\infty}^{\infty} e^{-\delta x^2} dx = \sqrt{\frac{\pi}{\delta}}$$

$\delta > 0 \quad a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-\delta(x-a)^2} \cdot 1\, dx = \int f(g(x)) \cdot g'(x)\, dx \qquad f(x) = e^{-\delta \cdot x^2} \quad g(x) = (x-a) \quad g'(x) = 1$$

$$\int_{g(a)}^{g(b)} e^{-\delta x^2} dx = \int_{-\infty}^{\infty} e^{-\delta x^2} dx = \sqrt{\frac{\pi}{\delta}} \qquad f(g(x)) = e^{-\delta(g(x))^2} = e^{-\delta(x-a)^2}$$

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(x)\, dx$$

Subst. Integral

$$\int_{-\infty}^{\infty} e^{-\delta(x-a)^2} dx = \sqrt{\frac{\pi}{\delta}}$$

$$\int_{-\infty}^{\infty} e^{-x^2 - 4x} dx = \int_{-\infty}^{\infty} e^{-(x^2+4x)} dx = \int_{-\infty}^{\infty} e^{-(x+2)^2 + 4} dx = e^4 \int_{-\infty}^{\infty} e^{-(x+2)^2} dx = e^4 \sqrt{\pi}$$

$$x^2 + 4x = x^2 + 4x + 4 - 4 = (x+2)^2 - 4$$

$$= \frac{f_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{x_0^2}\left(x^2 + ikx_0^2 x\right)} dx = \frac{f_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{x_0^2}\left(\left(x + \frac{ikx_0^2}{2}\right)^2 + \frac{k^2 x_0^4}{4}\right)} dx =$$

$$\underbrace{x^2 + ikx_0^2 \cdot x}_{2ab} = x^2 + ikx_0^2 \cdot x + \left(\frac{ikx_0^2}{2}\right)^2 - \left(\frac{ikx_0^2}{2}\right)^2 = \left(x + \frac{ikx_0^2}{2}\right)^2 + \frac{k^2 x_0^4}{4}$$

$$= \frac{f_0}{\sqrt{2\pi}} \cdot e^{-\frac{k^2 x_0^2}{4}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{x_0^2}\left(x + \frac{ikx_0^2}{2}\right)^2} dx = \frac{f_0}{\sqrt{2\pi}} e^{-\frac{k^2 x_0^2}{4}} \sqrt{\pi\, x_0^2} = \frac{f_0}{\sqrt{2}} \cdot e^{-\frac{k^2 x_0^2}{4}} \cdot x_0$$

$$\int_{-\infty}^{\infty} e^{-\delta(x-a)^2} dx = \sqrt{\frac{\pi}{\delta}}$$

Gauss-Integral

b) $$\tilde{f}(k) = \frac{f_0}{\sqrt{2}} e^{-\frac{k^2 x_0^2}{4}} \cdot x_0$$

Problem 2.1: Gaussian wave packet of a free particle

At initial time $t = 0$, a normalized wave packet of a free particle with mass m in one dimension is given as

$$\psi(x, 0) = \frac{1}{\sqrt{a\sqrt{\pi}}} \exp\left(-\frac{x^2}{2a^2} + ik_0 x\right).$$

The evolution of the wave packet is determined by the one dimensional Schrödinger equation for a free particle.

a) Calculate the Fourier transform of the initial wave packet $\psi(x, 0)$. (2 points)

$$\psi(k, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x, 0) e^{ikx} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{a\sqrt{\pi}}} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2a^2} + ik_0 x + ikx} dx =$$

$$-\frac{x^2}{2a^2} + ik_0 x + ikx = -\frac{x^2}{2a} + (k_0 + k) ix = -\frac{1}{2a}\left(x^2 - (k_0+k) \cdot i \cdot 2a \cdot x\right)$$

$$= -\frac{1}{2a}\left(\underbrace{x^2}_{a^2} - \underbrace{2a(k_0+k) i x}_{2ab} + (a(k_0+k) \cdot i)^2 - (a(k_0+k) \cdot i)^2\right)$$

$$= -\frac{1}{2a} \cdot \left((x - a(k_0+k) \cdot i)^2 + a^2 (k_0+k)^2\right)$$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-\frac{(k_0+k)^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{1}{2a}(x - a(\ldots))^2} dx$$

$$= \quad \| \cdot \sqrt{\pi \cdot 2a} =$$

miro