$A_1 \cup A_2 =$
$\{\omega \in \Omega \mid \omega \in A_1 \text{ oder } \omega \in A_2\}$

$A_1 \cap A_2 =$
$\{\omega \in \Omega \mid \omega \in A_1 \text{ und } \omega \in A_2\}$

i) $A_1 \cup A_2 \cup A_3$

ii) $(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c$

$\boxed{\text{De Morgan } (A \cup B)^c = A^c \cap B^c}$

iii) $(A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3) \cup (A_1^c \cap A_2^c \cap A_3^c)$

iv) $A_1^c \cup A_2^c \cup A_3^c$

v) [illegible]

vi) $(A_1 \cap A_2 \cap A_3^c) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1^c \cap A_2 \cap A_3)$

Hausaufgabe 1.2 (4 Punkte)

Beweisen Sie die folgenden Aussagen für einen Wahrscheinlichkeitsraum $(\Omega, \mathbb{P})$:

(i) Seien A und B zwei Ereignisse, für die gilt: $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$. Zeigen Sie, dass $\mathbb{P}(A \cap B) = \mathbb{P}(A^c \cap B^c)$.

(ii) Zeigen Sie, dass $\mathbb{P}(A \cap B) + \mathbb{P}(B \cap C) + \mathbb{P}(C \cap A) \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - 1$ für $A, B, C \subseteq \Omega$. Wann liegt Identität vor, d.h. wann gilt "="?

I) $\boxed{\mathbb{P}(A^c) = 1 - \mathbb{P}(A)}$

II) $A^c \cap B^c \overset{\text{De Morgan}}{=} (A \cup B)^c$

$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \text{ wenn } A \cap B = \emptyset}$

III) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$-(a+b-c+d)$
$= -a-b+c-d$

i) Beweis:

$$\mathbb{P}(A^c \cap B^c) \overset{II)}{=} \mathbb{P}((A \cup B)^c) \overset{I)}{=} \mathbb{P}((A \cup B)^c)$$
$$\overset{I}{=} 1 - \mathbb{P}(A \cup B) \overset{III}{=}$$
$$= 1 - (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B))$$
$$= 1 - \left(\frac{1}{2} + \frac{1}{2} - \mathbb{P}(A \cap B)\right) =$$
$$= 1 - (1 - \mathbb{P}(A \cap B)) =$$
$$= 1 - 1 + \mathbb{P}(A \cap B) =$$
$$\underline{\underline{\mathbb{P}(A \cap B)}}$$

(ii) Zeigen Sie, dass $\mathbb{P}(A \cap B) + \mathbb{P}(B \cap C) + \mathbb{P}(C \cap A) \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - 1$ für $A, B, C \subseteq \Omega$. Wann liegt Identität vor, d.h. wann gilt "="?

$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}$ I)

$\boxed{A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)}$ II) Assoziativgesetz

$\boxed{(A \cup B) \cap C = (A \cap C) \cup (B \cap C)}$ III) Distributivgesetz

Wir zeigen zunächst (Einschluss-Ausschluss-Formel)

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

Beweis:

$$\mathbb{P}(A \cup B \cup C) \overset{II}{=} \mathbb{P}((A \cup B) \cup C) \overset{I)}{=} \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C)$$
$$\overset{\text{Distributivgesetz}}{=} \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cap C) \cup (B \cap C))$$
$$\overset{I}{=} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(C) - \left[\mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \underbrace{\mathbb{P}((A \cap C) \cap (B \cap C))}_{\substack{= \mathbb{P}(A \cap C \cap B \cap C) \\ = \mathbb{P}(A \cap B \cap C)}}\right] =$$

$\boxed{C \cap C = C}$

$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

$$\mathbb{P}(A \cup B \cup C \cup D) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \ldots + \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap C \cap D) + \ldots - \mathbb{P}(A \cap B \cap C \cap D)$$

$\boxed{\text{Monotonie des W'maßes: } A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)}$

Einschluss-Ausschluss-Formel

$$1 = \mathbb{P}(\Omega) \geq \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \underbrace{\mathbb{P}(A \cap B \cap C)}_{\geq 0} \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C)$$

$$\Rightarrow 1 \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) \quad \Big| -1$$

$$0 \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) - 1 \quad \Big|$$

$$\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) \geq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - 1$$

Gleichheit gilt, wenn $\mathbb{P}(A \cup B \cup C) = 1$ und $\mathbb{P}(A \cap B \cap C) = 0$