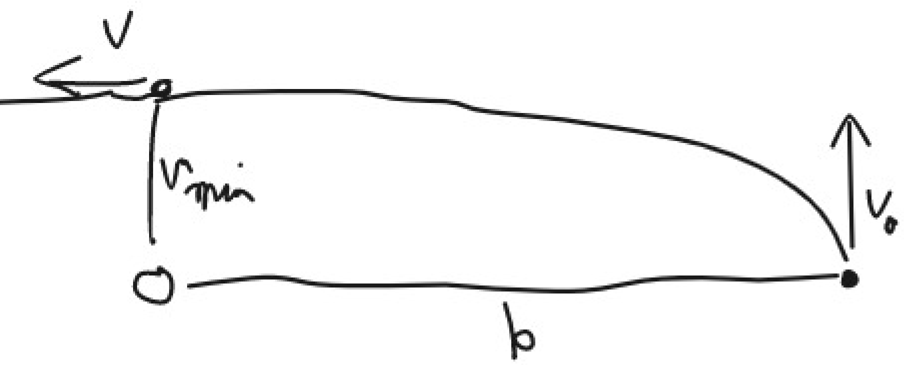
(v) (2 P) A particle with an initial velocity v_0 and mass m scatters in a central potential $U(r) = -\frac{k}{r^4}$. Use conversations laws to derive a relation between the impact parameter b and the minimal distance to the center r_{min}.

Energy and the is conserved

$$E = E_{kin} + E_{pot} =$$
$$= \frac{1}{2} m v^2 - \frac{k}{r^4} = \text{const}$$



$$L = m \cdot r \cdot v = \text{const}$$

$v =$ velocity at distance r_{min}

Unknowns: v, r_{min}

I) $m \cdot b \cdot v_0 = m \cdot r_{min} \cdot v \Rightarrow v = \frac{\cancel{m} \cdot b \cdot v_0}{\cancel{m} \cdot r_{min}} = \frac{b}{r_{min}} \cdot v_0$

II) $\frac{1}{2} m v_0^2 - \frac{k}{b^4} = \frac{1}{2} m v^2 - \frac{k}{r_{min}^4}$

v in II)

$$\underbrace{\frac{1}{2} m v_0^2 - \frac{k}{b^4}}_{=:q} = \frac{1}{2} m \frac{b^2}{r_{min}^2} \cdot v_0^2 - \frac{k}{r_{min}^4} \quad \Big| \cdot r_{min}^4$$

$$q \cdot r_{min}^4 = \underbrace{\frac{1}{2} m b^2 v_0^2}_{=:c} \cdot r_{min}^2 - k$$

$$q \cdot r^4 - c \cdot r^2 + k = 0$$

$z := r^2$
$$q \cdot z^2 - c \cdot z + k = 0$$

Problem 3 - (total 15.0 P)

For a test particle of mass m we will consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - 1 + \frac{2M}{r} + \frac{1}{2}\left(\frac{r^3}{r-2M}\right)\dot{\phi}^2 \qquad (27)$$

Here polar coordinates have been used and M is a constant.

(i) *(1 P)* What are the conserved quantities of this system? You do not have to explicitly compute them at this point.

$E =$ Energy is conserved because Lagranian doesn't depend explicitly on time

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt}\left[\underbrace{\frac{r^3}{r-2M}\cdot\dot{\phi}}_{=:\,p_\phi}\right] = 0$$

$$\underline{L = p_\phi = \text{const}}$$

(ii) (*3.5 P*) It can be shown, that the total energy E of the system is given by:

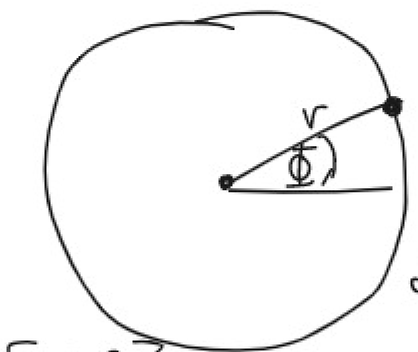
$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) = \frac{1}{2}m\dot{r}^2 + \underbrace{1 - \frac{2M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}}_{V_{eff}}. \qquad (28)$$

Here L is one of the previously found conserved quantities.

Show that for:

$$\Rightarrow r_\pm = L^2\left(\frac{1 \pm \sqrt{1 - \frac{24M^2}{L^2}}}{4M}\right) \qquad (29)$$

there exist circular orbits in the effective potential $V_{eff}(r)$. Analyse the solutions given in (29) with regards to their dependence on L and M. *Hint: You should consider three different cases.*



$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - V_{eff}(r)$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{d}{dt}\left[m\dot{r}\right] = \frac{\partial V}{\partial r}$$

$$m\ddot{r} = \frac{\partial V}{\partial r}$$

circular orbit $= r = \text{const} \Rightarrow \dot{r} = 0$

$$\boxed{\frac{\partial V}{\partial r} = 0 \quad \text{for circular orbits}}$$

$$\frac{\partial V_{eff}}{\partial r} = \frac{2M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} \overset{!}{=} 0 \quad \Big| \cdot r^4$$

$$2Mr^2 - L^2 \cdot r + 3ML^2 = 0$$

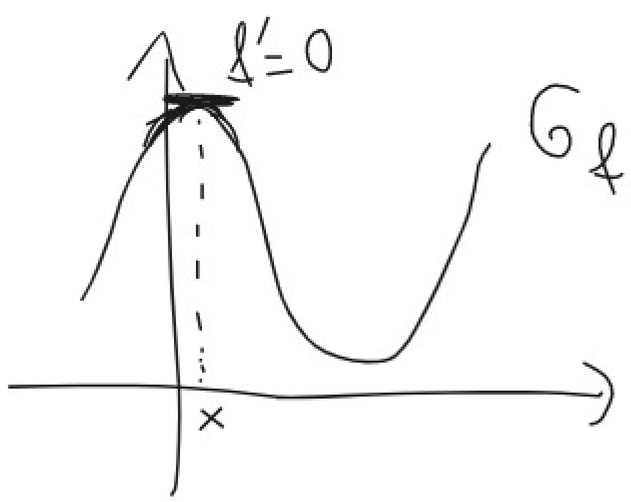
$$ax^2 + bx + c = 0$$

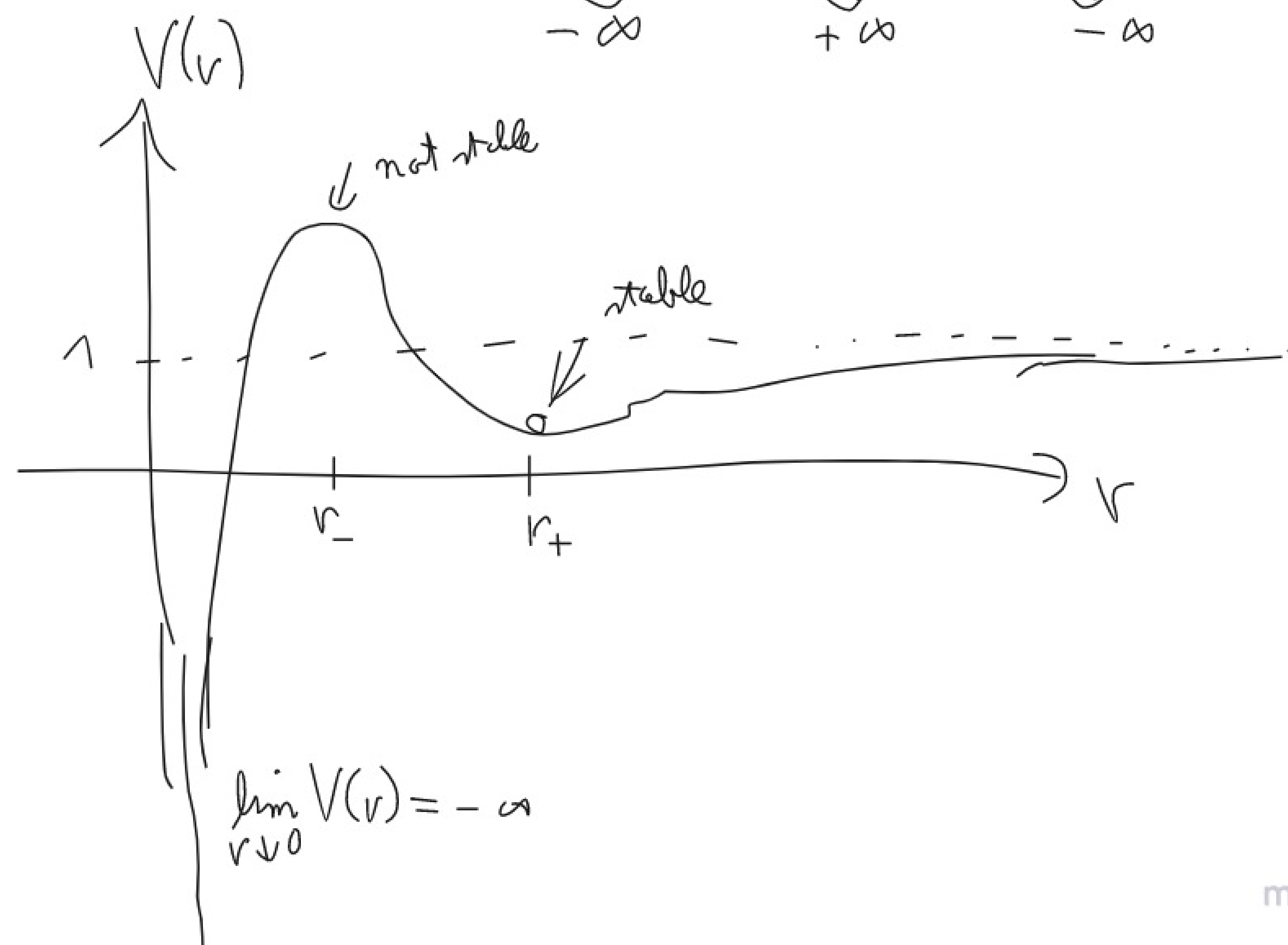
$$x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(iii) (*3.0 P*) Perform a stability analysis of the previously derived circular orbits. *Hint: Analyse the potential and sketch it instead of performing a lengthy computation.*

$$\lim_{r \to \infty} V(r) = \lim_{r \to \infty} 1 - \underbrace{\frac{2M}{r}}_{0} + \underbrace{\frac{L^2}{2r^2}}_{0} - \underbrace{\frac{ML^2}{r^3}}_{0} = 1$$

$$\lim_{r \downarrow 0} V(r) = \lim_{r \to 0} 1 - \underbrace{\frac{2M}{r}}_{-\infty} + \underbrace{\frac{L^2}{2r^2}}_{+\infty} - \underbrace{\frac{ML^2}{r^3}}_{-\infty} = -\infty$$





(iv) (*1 P*) Does there exist an angular momentum barrier? What does this tell you with regards to the completeness of the potential?

Angular momentum $= L = m \cdot r^2 \cdot \dot{\phi} = m \cdot r^2 \cdot L_\phi \frac{r-2M}{r^3} = m \cdot \frac{L_\phi (r-2M)}{r} \xrightarrow{r \to 0} -\infty$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L_\phi = \frac{r^3}{r-2M} \cdot \dot{\phi} \Rightarrow \dot{\phi} = L_\phi \frac{r-2M}{r^3} = L_\phi$$

Taylor

$$\boxed{f(h + x_0) \approx f(x_0) + \frac{f'(x_0)}{1!} \cdot h + \frac{f''(x_0)}{2!} \cdot h^2}$$

(vi) (*1.5 P*) Determine and solve the equation of motion for r.

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) = \frac{1}{2}m\dot{r}^2 + 1 - \frac{2M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}.$$

$(\times -1)$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - V_{eff}$$

Taylor

$$V_{eff}(r_0 + r) \overset{\Downarrow}{=} V(r_0) + \overbrace{V'(r_0)}^{=0} \cdot r + \underbrace{\frac{V''(r_0)}{2}}_{=:k} \cdot r^2$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = \frac{\partial \mathcal{L}}{\partial r}$$

$$m\ddot{r} = \frac{\partial}{\partial r}\left(-\left(V(r_0) + 0 + k \cdot r^2\right)\right)$$

$$m\ddot{r} = -2kr \quad | : m$$

$$\ddot{r} = -\underbrace{\frac{2k}{m}}_{\omega^2} \cdot r \qquad \omega = \sqrt{\frac{2k}{m}}$$

Harmonic osszilator

$$\ddot{x} = -\omega^2 \cdot x$$

$$x(t) = A \cdot \cos(\omega t) + B \cdot \sin(\omega t)$$