$X_n \to X$ a.s. $:\Leftrightarrow$
$\mathbb{P}(\{\omega\in\Omega : X_n(\omega) \xrightarrow{n\to\infty} X(\omega)\}) = 1$

$X_n \to X$ pointwise
$\forall \omega\in\Omega: X_n(\omega) \xrightarrow{n\to\infty} X(\omega)$

$X_n \to_p X \;:\Leftrightarrow$
$\forall \varepsilon>0: \mathbb{P}(|X_n - X|>\varepsilon) \xrightarrow{n\to\infty} 0$

9. Use the definition of expected value in terms of simple variables to prove that if $X \geq 0$ and $E|X| = 0$ then $X = 0$ almost surely.

X simple, if $|X(\Omega)| = |\{X(\omega) \mid \omega\in\Omega\}| < \infty$

$\mathbb{E}[X] = \sup_{\substack{Y \text{ simple} \\ Y\leq X}} \mathbb{E}[Y]$

$X=5$ a.s $\Leftrightarrow$
$\mathbb{P}(\{\omega\in\Omega \mid X(\omega)=5\}) = 1$

$X\geq Y$ a.s. $\Leftrightarrow$
$\mathbb{P}(\{\omega\in\Omega \mid X(\omega)\geq Y(\omega)\}) = 1$

Let $X\geq 0$ and $\mathbb{E}[X]=0$.
To show: $X=0$ a.s, i.e.
$\mathbb{P}(X=0) = \mathbb{P}(\{\omega\in\Omega \mid X(\omega)=0\}) = 1$

Suppose for contradiction that
$\mathbb{P}(X=0) < 1$

$\underbrace{\mathbb{P}(A)} + \mathbb{P}(A^c) = 1$
$\mathbb{P}(A^c) = 1 - \underbrace{\mathbb{P}(A)}_{<1}$
$> 1-1 = 0$

$\Rightarrow \mathbb{P}(X>0) > 0$
$A_n := \{X > \frac{1}{n}\}$

$\bigcup_{n\in\mathbb{N}} A_n = \{X>0\}$ is clear

$A_n \nearrow \{X>0\} =: A$
$\{X>0\} = \{\omega\in\Omega \mid X(\omega)>0\}$

Continuity of probability measure from below:
$(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $A_1\subseteq A_2 \subseteq A_3 \subseteq \dots$, i.e. $A_n \xrightarrow{n\to\infty} \bigcup_{n\in\mathbb{N}} A_n =: A$
$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{n\in\mathbb{N}} A_n) = \mathbb{P}(A)$

$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\underset{\{X>0\}}{A}) > 0$

$(a_n)_n \subseteq \mathbb{R}$, $a_n \xrightarrow{n\to\infty} a > 0$
$\Rightarrow \exists N\in\mathbb{N}$ such that $a_N > 0$

$\Rightarrow$ We find N large enough s.t.
$\mathbb{P}(A_N) > 0$

$\Rightarrow \mathbb{P}(\{X > \frac{1}{N}\}) > 0$

$Y := \frac{1}{N} \cdot 1\!\!1_{X>\frac{1}{N}}$ $\quad \cdot\, Y\leq X$ ✓
$\cdot\, |Y(\Omega)| = |\{\frac{1}{N}, 0\}| = 2 < \infty$ ✓

$Y\leq X$
$|Y(\Omega)| < \infty$
$\mathbb{E}[Y] > 0$

$\mathbb{E}[X] = \sup_{\substack{Y\leq X \\ Y \text{ simple}}} \mathbb{E}[Y]$

$\mathbb{E}[Y] = \underbrace{0\cdot \mathbb{P}(Y=0)}_{=0} + \frac{1}{N}\cdot \mathbb{P}(Y=\frac{1}{N}) = \frac{1}{N}\cdot \underbrace{\mathbb{P}(\{X>\frac{1}{N}\})}_{>0} > 0$

$\Rightarrow \mathbb{E}[X] = \sup_{\substack{Z\leq X \\ Z \text{ simple}}} \mathbb{E}[Z] \geq \mathbb{E}[Y] > 0$

Contradiction to
$\mathbb{E}[X] = 0$!

$\Rightarrow X = 0$ a.s.

$X\in\{1,2,3\}$
$\mathbb{E}[X] = 1\cdot \mathbb{P}(X=1)$
$+ 2\cdot \mathbb{P}(X=2)$
$+ 3\cdot \mathbb{P}(X=3)$

Exercise: Find $(X_n)_{n\in\mathbb{N}}$, X random variables on same space $(\Omega, \mathcal{F}, \mathbb{P})$ such that
$X_n \xrightarrow{n\to\infty}_p X$ but not $X_n \xrightarrow{\text{a.s.}} X$

Borel-σ-field

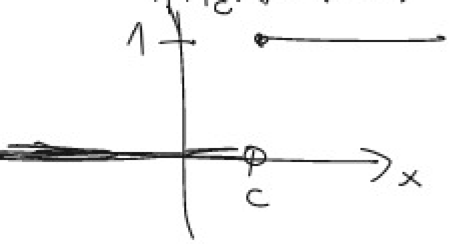
$\Omega = [0,1]$ $\quad \mathcal{F} = \mathcal{B}([0,1])$ $\quad \mathbb{P} = \lambda|_{[0,1]}$
Lebesgue-measure

$X := 0$
$X_1 = 1\!\!1_{[0,1]}$
$X_2 = 1\!\!1_{[0,\frac{1}{2}]}$ $\quad X_3 = 1\!\!1_{[\frac{1}{2},1]}$
$X_4 = 1\!\!1_{[0,\frac{1}{3}]}$ $\quad X_5 = 1\!\!1_{[\frac{1}{3},\frac{2}{3}]}$ $\quad X_6 = 1\!\!1_{[\frac{2}{3},1]}$
$X_5 = 1\!\!1_{[0,\frac{1}{4}]}$ $\quad X_6 = 1\!\!1_{[\frac{1}{4},\frac{2}{4}]}$ and so on

$\Rightarrow X_n \to_p 0$ but
$\mathbb{P}(\{\omega\in\Omega \mid X_n(\omega) \to \overset{0}{X(\omega)}\}) = 0$
$\Rightarrow \neg(X_n \xrightarrow{n\to\infty} X$ a.s.$)$

$X_n \to_d X :\Leftrightarrow$ $\forall x$ with F_X is continuous in x holds: $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

$F_c(x) = \mathbb{P}(c\leq x)$



10. Show that if $X_n \longrightarrow_d c$ then $X_n \longrightarrow_p c$.

$X_n \to_d c$, i.e. $\forall x\in\mathbb{R}\setminus\{c\}: \lim_{n\to\infty} F_{X_n}(x) = F_c(x)$

To show: $X_n \to_p c$, i.e.

miro