Übung 2.9 (Martingale bei der Irrfahrt mit Gaußschen Zuwächsen) *Es seien $(Z_n)_{n\in\mathbb{N}}$ i.i.d. normalverteilte Zufallsvariablen auf einem Wahrscheinlichkeitsraum $(\Omega, \mathcal{A}, P)$ mit Erwartung $E_P[Z_1] = 0$ und Standardabweichung $\sigma_P(Z_1) = \sigma > 0$. Weiter sei $(X_n)_{n\in\mathbb{N}_0}$ die Irrfahrt mit Start in $X_0 = 0$ und Zuwächsen Z_n:*

$$X_n = X_{n-1} + Z_n, \quad (n \in \mathbb{N}). \tag{288}$$

Es sei $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ die natürliche Filtration dazu:

$$\mathcal{F}_n = \sigma(X_m : m \in \mathbb{N}_0, m \le n) = \sigma(Z_m : m \le n)$$

Zeigen Sie, dass die folgenden Prozesse $\mathcal{F}$-Martingale sind:

1. $A_n = X_n^2 - n\sigma^2$, $(n \in \mathbb{N}_0)$,
2. $B_n = X_n^4 - 6X_n^2 n\sigma^2 + 3n^2\sigma^4$, $(n \in \mathbb{N}_0)$,
3. $C_n = \exp\left(X_n - \frac{n\sigma^2}{2}\right)$, $(n \in \mathbb{N}_0)$.

1.) To show: $(A_n)_{n\in\mathbb{N}_0}$ martingale, i.e.

1) $\mathbb{E}[|A_n|] < \infty \quad \forall n \in \mathbb{N}_0$

2) A_n is $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N}_0$

3) $\mathbb{E}[A_{n+1} \mid \mathcal{F}_n] = A_n$

zu 2) $A_n = X_n^2 - n\cdot\sigma^2$ is $\mathcal{F}_n$-measurable, because X_n is $\mathcal{F}_n$-measurable and because sums and product of $\mathcal{F}_n$-measurable random variables are $\mathcal{F}_n$-measurable

$\mathcal{F}_n = \sigma(X_m \mid m \le n) = \sigma(X_1, X_2, \dots, X_n)$

zu 3) $\mathbb{E}[A_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[X_{n+1}^2 - (n+1)\cdot\sigma^2 \mid \mathcal{F}_n\right]$

Lemma 1.28 (Linearität der bedingten Erwartung – Version für nichtnegative Zufallsvariablen) *Es seien $X, Y : \Omega \to [0, \infty]$ zwei nichtnegative Zufallsvariablen auf einem Wahrscheinlichkeitsraum $(\Omega, \mathcal{A}, P)$ und $\mathcal{F} \subseteq \mathcal{A}$ eine Unter-σ-Algebra. Weiter sei $a \in [0, \infty]$. Dann gilt:*

1. $E[X + Y|\mathcal{F}] = E[X|\mathcal{F}] + E[Y|\mathcal{F}]$ *P-fast sicher.*
2. $E[aX|\mathcal{F}] = aE[X|\mathcal{F}]$ *P-fast sicher.*

$$= \mathbb{E}\left[(X_n + Z_{n+1})^2 - (n+1)\cdot\sigma^2 \mid \mathcal{F}_n\right]$$

$$= \mathbb{E}\left[X_n^2 + 2X_n Z_{n+1} + Z_{n+1}^2 - (n+1)\cdot\sigma^2 \mid \mathcal{F}_n\right]$$

linearity of conditional expectation

$$= \mathbb{E}[X_n^2 \mid \mathcal{F}_n] + 2\,\mathbb{E}[X_n Z_{n+1} \mid \mathcal{F}_n] + \mathbb{E}[Z_{n+1}^2 \mid \mathcal{F}_n] - (n+1)\sigma^2$$

$(Z_m)_{m\in\mathbb{N}}$ independent

$\boxed{\begin{array}{l} X, Y \text{ RV} \\ X\ \mathcal{F}\text{-measurable} \\ \mathbb{E}[X\cdot Y \mid \mathcal{F}] = X\cdot\mathbb{E}[Y \mid \mathcal{F}] \end{array}}$

X_n $\mathcal{F}_n$-measurable

$$= X_n^2 + 2\cdot X_n\cdot\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] + \mathbb{E}[Z_{n+1}^2 \mid \mathcal{F}_n] - (n+1)\sigma^2$$

$\mathcal{F}_n = \sigma(Z_m \mid m \le n)$

$\boxed{\begin{array}{l} X \text{ independent of } \mathcal{F} \\ \mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X] \end{array}}$

Z_{n+1} is independent of $\mathcal{F}_n$

$$= X_n^2 + 2\,X_n\cdot\underbrace{\mathbb{E}[Z_{n+1}]}_{=\mathbb{E}[Z_1]=0} + \underbrace{\mathbb{E}[Z_{n+1}^2]}_{=\mathbb{E}[Z_{n+1}^2] - \mathbb{E}[Z_{n+1}]^2 = \mathrm{Var}(Z_{n+1}) = \sigma^2} - (n+1)\sigma^2 =$$

$\mathbb{E}[Z_{n+1}] = 0$

$$= X_n^2 + \sigma^2 - (n+1)\sigma^2 =$$

$$= X_n^2 - n\sigma^2 = A_n$$

zu 1) $\mathbb{E}[|A_n|] =$

$$= \mathbb{E}\left[|X_n^2 - n\sigma^2|\right] \overset{\triangle\text{-inequality}}{\le} \mathbb{E}[X_n^2] + n\sigma^2$$

$$\le \mathbb{E}\Big[\big(\underbrace{Z_1 + \dots + Z_n}_{\sim\mathcal{N}(0,\, n\cdot\sigma^2)}\big)^2\Big] + n\sigma^2$$

$$= n\cdot\sigma^2 + n\sigma^2 = 2n\sigma^2 < \infty$$

[illegible]

2) To show: $(B_n)_{n\in\mathbb{N}_0}$ martingale, i.e.

1) $\mathbb{E}[|B_n|] < \infty \quad \forall n \in \mathbb{N}_0$

2) B_n is $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N}_0$

3) $\mathbb{E}[B_{n+1} \mid \mathcal{F}_n] = B_n \quad \forall n \in \mathbb{N}_0$

zu 2) $B_n = X_n^4 - 6X_n^2 n\sigma^2 + 3n^2\sigma^4$ is $\mathcal{F}_n$-measurable, $\leftarrow g(X_1, \dots, X_n)$

because X_n is $\mathcal{F}_n$-measurable and sums and product of $\mathcal{F}_n$-measurable RV's are $\mathcal{F}_n$-measurable

$(a+b)^2 = a^2 + 2ab + b^2$

$$\boxed{\begin{array}{l}\text{binomial formula}\\ (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\end{array}}$$

linearity
$\mathbb{E}[X + aY \mid \mathcal{F}] = \mathbb{E}[X \mid \mathcal{F}] + a\,\mathbb{E}[Y \mid \mathcal{F}]$

X $\mathcal{F}$-measurable
$\mathbb{E}[X \cdot Y \mid \mathcal{F}] = X \cdot \mathbb{E}[Y \mid \mathcal{F}]$

X independent of $\mathcal{F}$
$\mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X]$

zu 3) $\mathbb{E}[B_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[X_{n+1}^4 - 6X_{n+1}^2 (n+1)\sigma^2 + 3(n+1)^2\sigma^4 \mid \mathcal{F}_n\right]$

$= \mathbb{E}[X_{n+1}^4 \mid \mathcal{F}_n] - 6\cdot(n+1)\cdot\sigma^2\cdot\mathbb{E}[X_{n+1}^2 \mid \mathcal{F}_n] + 3\cdot(n+1)^2\sigma^4$

$= \mathbb{E}[(X_n + Z_{n+1})^4 \mid \mathcal{F}_n] - 6\cdot(n+1)\cdot\sigma^2\,\mathbb{E}[(X_n + Z_{n+1})^2 \mid \mathcal{F}_n] + 3(n+1)^2\sigma^4$

$= \mathbb{E}\Big[X_n^4 + \binom{4}{1} X_n^3 Z_{n+1} + \underbrace{\binom{4}{2}}_{=\frac{4!}{2!(4-2)!} = \frac{4!}{2\cdot 2} = 6} X_n^2 Z_{n+1}^2 + \underbrace{\binom{4}{3}}_{=4} X_n Z_{n+1}^3 + Z_{n+1}^4 \;\Big|\; \mathcal{F}_n\Big]$

$\quad - 6\cdot(n+1)\cdot\sigma^2\cdot\mathbb{E}[X_n^2 + 2X_nZ_{n+1} + Z_{n+1}^2 \mid \mathcal{F}_n] + 3\cdot(n+1)^2\cdot\sigma^2$

$= X_n^4 + 4\cdot X_n^3 \underbrace{\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]}_{=\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_1] = 0} + 6\cdot X_n^2\cdot \underbrace{\mathbb{E}[Z_{n+1}^2 \mid \mathcal{F}_n]}_{=\mathbb{E}[Z_{n+1}^2] = \mathbb{E}[Z_{n+1}^2] - \mathbb{E}[Z_{n+1}]^2 = \mathrm{Var}(Z_{n+1}) = \mathrm{Var}(Z_1) = \sigma^2} + 4\cdot X_n\cdot \underbrace{\mathbb{E}[Z_{n+1}^3 \mid \mathcal{F}_n]}_{=\mathbb{E}[Z_{n+1}^3] = \mathbb{E}[Z_1^3] = 0} + \mathbb{E}[Z_{n+1}^4 \mid \mathcal{F}_n]$

$\quad - 6\cdot(n+1)\,\sigma^2\left(X_n^2 + 2\cdot X_n \underbrace{\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]}_{=\mathbb{E}[Z_{n+1}] = 0} + \sigma^2\right) + 3\cdot(n+1)^2\cdot\sigma^2$

$\mathbb{E}[Z_{n+1}^4] = \mathbb{E}[Z_{n+1}^4 \mid \mathcal{F}_n]$

$= \int_{-\infty}^{\infty} x^4 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}\, dx =$

$$\int_a^b f\, g'\, dx = [f\cdot g]_a^b - \int_a^b f' g\, dx$$

$$\left(-\sigma^2 e^{-\frac{x^2}{2\sigma^2}}\right)' = -\frac{\cancel{2}x}{\cancel{2}\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

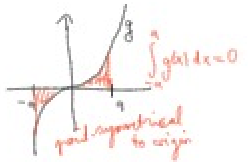
$$\cancel{= -\sigma^2 \cdot \frac{1}{\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

$$\int_{-\infty}^{\infty} \underbrace{x^3}_{f} \cdot \underbrace{x e^{-\frac{x^2}{2\sigma^2}}}_{g'}\, dx = \underbrace{\left[x^3 \cdot (-\sigma^2) \cdot e^{-\frac{x^2}{2\sigma^2}}\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} 3x^2 \cdot (-\sigma^2)\, e^{-\frac{x^2}{2\sigma^2}}\, dx$$

$$= 3\sigma^2 \cdot \sqrt{2\pi\sigma^2} \cdot \underbrace{\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}\, dx}_{= \mathrm{Var}(Z_1) = \mathbb{E}[Z_1^2] = \sigma^2} = 3\cdot(\sigma^2)^2 \cdot \sqrt{2\pi\sigma^2}$$

$$\mathbb{E}[Z_1^3] = \int_{-\infty}^{\infty} x^3 f_{Z_1}(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^3 \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\, dx = 0$$



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