3. The one-dimensional simple random walk is the Markov chain $X_n, n \geq 0$, whose states are all the integers, and which has the transition probabilities

$$P_{i,i+1} = 1 - P_{i,i-1} = p$$

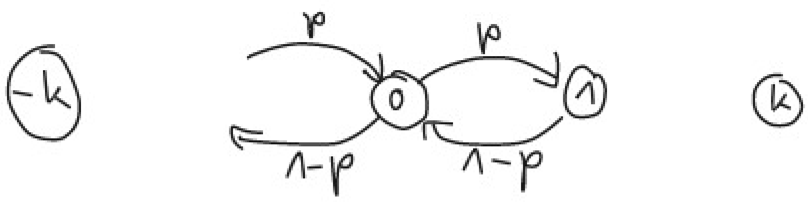
Show that this chain is recurrent when $p = 1/2$, and transient for all $p \neq 1/2$. When $p = 1/2$, the chain is called the 1-dimensional simple symmetric random walk.

$$T_k = \inf\{n \in \mathbb{N} \mid X_n \in \{-k, k\}\}$$

$X_{T_1} \in \{1, -1\}$

$X_{T_2} \in \{2, -2\}$

$X_{T_k} \in \{-k, k\}$



$\forall k \in \mathbb{N}: \quad T_k < \infty \quad$ a.s.

Proof:

<u>To show:</u> for $p = \frac{1}{2}$ $\quad \mathbb{P}(\exists i \in \mathbb{N}: X_i = 0 \mid X_0 = 0) \overset{!}{=} 1$, i.e.

$$\mathbb{P}(\forall i \in \mathbb{N}: X_i \neq 0 \mid X_0 = 0) = 0$$

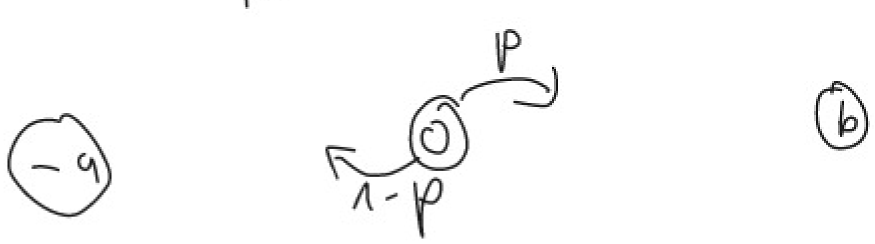
$$\overset{\parallel}{\tfrac{1}{2} \cdot \mathbb{P}(X_{T_1} = 1) \cdot \overset{\frac{1}{2}}{\overset{\parallel}{\mathbb{P}(X_{T_2} = 2)}} \cdot \mathbb{P}(X_{T_4} = 4) \cdot \mathbb{P}(X_{T_8} = 8) \cdot \mathbb{P}(X_{T_{16}} = 16) \cdots}$$

$$+ \tfrac{1}{2} \cdot \mathbb{P}(X_{T_1} = -1) \cdot \mathbb{P}(X_{T_2} = -2) \cdot \mathbb{P}(X_{T_4} = -4) \cdots$$

$$= \left(\tfrac{1}{2}\right)^{\infty} + \left(\tfrac{1}{2}\right)^{\infty} = 0$$

Let $p > \frac{1}{2}$.

$a, b \in \mathbb{N}$. Let $T_{a,b} := \inf\{n \in \mathbb{N} : X_n \in \{-a, b\}\}$



$T_{a,b} < \infty \quad$ a.s. ?

What is $\mathbb{P}(X_{T_{a,b}} = -a)$ and $\mathbb{P}(X_{T_{a,b}} = b)$?

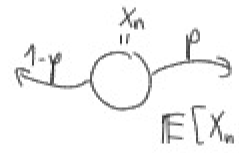
$(X_n)_{n \in \mathbb{N}} \qquad (\mathcal{F}_n)_{n \in \mathbb{N}} = (\sigma(X_1, \dots, X_n))_{n \in \mathbb{N}}$

Dann heißt X ein *Martingal* (bezüglich $\mathcal{F}$), wenn

$$\mathrm{E}(X_{n+1}|\mathcal{F}_n) = X_n \quad P\text{-fast sicher für alle } n \in \mathbb{N}$$

gilt.

$(\Rightarrow \mathbb{E}[X_n] = \mathbb{E}[X_0] \;\forall n \in \mathbb{N}$ and $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ $\forall$ stopping times T with $T<\infty$ a.s.$)$



$$\mathbb{E}[X_{n+1}|X_n] = X_n$$
$$= \mathbb{E}[X_{n+1}|\mathcal{F}_n]$$

$X_n = \sum_{i=1}^{n} Y_i \qquad Y_i \in \{-1,1\}$, i.i.d.

$X_{n+1} = \sum_{i=1}^{n+1} Y_i = X_n + \qquad \mathbb{P}(Y_1 = 1) = p = 1 - \mathbb{P}(Y_1 = 0)$

We look for $\delta \in \mathbb{R}$ s.t. that $(e^{\delta X_n})_{n\in\mathbb{N}} = (e^{\delta \sum_{i=1}^{n} Y_i})_{n\in\mathbb{N}}$ is a martingale.

$$\mathbb{E}[e^{\delta X_{n+1}} \mid \mathcal{F}_n] = \mathbb{E}[e^{\delta(X_n + Y_{n+1})} \mid \mathcal{F}_n] =$$
$$= \mathbb{E}[e^{\delta X_n} \cdot e^{\delta Y_{n+1}} \mid \mathcal{F}_n] =$$

$e^{\delta X_n}$ is $\mathcal{F}_n$-measurable

$$= e^{\delta X_n} \cdot \mathbb{E}[e^{\delta \cdot Y_{n+1}} \mid \mathcal{F}_n] =$$

Y_{n+1} is independent of $\mathcal{F}_n$

$$= e^{\delta X_n} \; \mathbb{E}[e^{\delta \cdot Y_{n+1}}]$$
$$\overset{!}{=} e^{\delta \cdot X_n} \cdot 1$$

must be martingale

To show:

$$\mathbb{E}[e^{\delta \cdot Y_{n+1}}] \overset{!}{=} 1$$
$$e^{\delta} \cdot p + e^{-\delta} \cdot (1-p) \overset{!}{=} 1$$
$$\Rightarrow p e^{\delta} + e^{-\delta}(1-p) - 1 = 0$$

We find a solution $\delta \neq 0$.

$\mathbb{E}[Y_{n+1}^2] =$
$= 1^2 \cdot p + (-1)^2 \cdot (1-p)$
$\mathbb{E}[\sin(Y_{n+1})]$
$= \sin(1) \cdot p + \sin(-1) \cdot (1-p)$

$X_{T_{a,b}} \in \{-a, b\}$

martingale

$$\underbrace{\mathbb{E}[e^{\delta \cdot X_{T_{a,b}}}]}_{} = \underbrace{\mathbb{E}[e^{\delta \cdot X_0}]}_{= 1}, \quad X_0 = 0$$

I) $e^{\delta \cdot (-a)} \cdot \mathbb{P}(X_{T_{a,b}} = -a) + e^{\delta \cdot b} \cdot \mathbb{P}(X_{T_{a,b}} = b) = 1$, with $p_1 = \mathbb{P}(X_{T_{a,b}} = -a)$, $p_2 = \mathbb{P}(X_{T_{a,b}} = b)$

II) $\mathbb{P}(X_{T_{a,b}} = -a) + \mathbb{P}(X_{T_{a,b}} = b) = 1$

$\Rightarrow$ You can calculate a and b

I) $e^{-\delta a} \cdot p_1 + e^{\delta b} \cdot p_2 = 1$

II) $p_1 + p_2 = 1 \Rightarrow p_1 = 1 - p_2$

$e^{-\delta a}(1-p_2) + e^{\delta b} p_2 = 1$

$p_2 = \dfrac{1 - e^{-\delta a}}{-e^{-\delta a} + e^{\delta b}}$

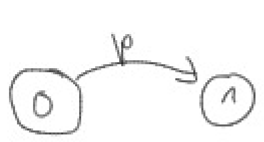
$x = e^{\delta a}$

$p_1 = \dfrac{e^{\delta b} - 1}{-e^{-\delta a} + e^{\delta b}} \underset{a=b}{=} \dfrac{e^{\delta a} - 1}{e^{\delta a} - e^{-\delta a}} = \dfrac{x-1}{x - \frac{1}{x}} =$

$= \dfrac{x-1}{\frac{x^2-1}{x}} = \dfrac{\cancel{(x-1)} \cdot x}{\cancel{(x-1)} \cdot (x+1)} =$

$= \dfrac{x}{x+1} = \dfrac{1}{\frac{x+1}{x}} = \dfrac{1}{1 + \frac{1}{x}} = \dfrac{1}{1 + e^{-\delta a}}$

$p_2 = \dfrac{e^{-\delta a}}{1 + e^{-\delta a}} = \dfrac{1}{e^{\delta a} + 1}$



$$\mathbb{P}(\forall i \in \mathbb{N}: X_i \neq 0 \mid X_0 = 0) =$$
$$= p \cdot \mathbb{P}(X_{T_1} = 1) \cdot \mathbb{P}(X_{T_2} = 2) \cdot \mathbb{P}(X_{T_4} = 4) \cdot \ldots$$
$$+ (1-p) \cdot \mathbb{P}(X_{T_1} = -1) \cdot \mathbb{P}(X_{T_2} = -2) \cdot$$

$T_1 = T_{1,1}$

$$= p \cdot \frac{1}{1+e^{\delta}} \cdot \frac{1}{1+e^{2\delta}} \cdot \frac{1}{1+e^{4\delta}} \cdot \ldots$$
$$+ (1-p) \cdot \frac{1}{1+e^{-\delta}} \cdot \frac{1}{1+e^{-2\delta}} \cdot \frac{1}{1+e^{-4\delta}} \cdot$$

$$= p \cdot \prod_{i=1}^{\infty} \frac{1}{1+e^{\delta \cdot i}}$$
$$+ (1-p) \cdot \prod_{i=1}^{\infty} \frac{1}{1+e^{-\delta i}} > 0$$

To show: $\displaystyle\prod_{i=1}^{\infty} \frac{1}{1+e^{\delta i}} > 0$ or $\displaystyle\prod_{i=1}^{\infty} \frac{1}{1+e^{-\delta i}} > 0$

$$\Updownarrow$$

$$\ln\left(\prod_{i=1}^{\infty} \ldots\right) \neq -\infty$$

$\ln(a \cdot b) =$
$= \ln(a) + \ln(b)$

$$= \sum_{i=1}^{\infty} \ln\left(\frac{1}{1+e^{\delta i}}\right) = \sum_{i=1}^{\infty} (-1) \cdot \ln(1 + e^{\delta i}) \neq -\infty$$

$\ln(a^b) =$
$= b \cdot \ln(a)$

$\frac{1}{x} = x^{-1}$

$$\Updownarrow$$

$$\sum_{i=1}^{\infty} \ln(1 + e^{\delta i}) \neq \infty$$

if $\delta < 0$